FREE CONVECTION AND BIFURCATION

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The occurence of secondary stationary flows was analyzed in [1 to 4] for a number of problems of hydrodynamics, by means of the topological method, and the application of the Krasnosel'skii's bifurcation theorem [5]. This method, although very general and requiring only a minimum of "outset information", does not, however, afford the possibility of investigating the spectrum distribution, or determining the number of occurring solutions.

The most detailed information on bifurcation can be obtained by the analytical method of Liapunov-Schmidt. This, not only yields qualitative results, but is also an effective tool for computing secondary flows in the range of problems considered here. The main difficulty encountered in applications of this method centers around the solution of linearized problems. Genrally speaking, such problems have to be solved numerically, although there are cases in which quantitative results may be obtained independently of computations. Such instances were analyzed in [1 to 4].

It should be pointed out that a combination of topological and analytical methods yields the most finalized and full results : in particular, a complete picture of stability loss in a convection problem may be obtained in this way. It is shown in this paper that two secondary flows appear, immediately after the loss of stability (and the problem has no other nontrivial solutions). This takes place in the case in which the first eigen number of the linearized problem is a prime number. Several examples are adduced in which the condition of primeness is verified, viz. convection in a horizontal layer, and in a vertical cylindrical vessel of considerable height. These results are set out in Section 2. It will be subsequently shown that both secondary flows are stable. It should be noted that in the case of a layer (as well as in certain other cases) the bifurcation on transition through subsequent critical numbers proceeds similarly, but gives rise to unstable solutions.

Theorem 1.1 required in the subsequent analysis will be proved in Section 1, in which the Liapunov-Schmidt method is applied to a case which, although special, is frequently met with in mathematical physics problems. A multiple spectrum is also admissible here. We note that this theorem has made it now possible to establish that the Taylor secondary flow between rotating cylinders is uniquely defined (with an accuracy of the order of shear along the tube axis).

The Liapunov-Schmidt method yields the most detailed information on the nature of bifurcation (number of solutions, spectrum distribution, etc.). This method requires, on the other hand, much more information about operators than the topological method.

The multiple spectrum causes particular complications. It is sometimes possible to reduce the problem to a simple spectrum by looking for a solution from a particular subspace (for example, by imposing in hydrodynamic problems certain conditions of evenness on the unknown functions, as was done in [1 to 4].

There still remains unresolved the question of other solutions. It will be shown in the first Section how this is resolved in one case, in which the spectrum multiplicity is the result of the invariant character of the problem with respect to a particular group of transformations. It appears that, subject to certain conditions (see Theorem 1.1), all solutions may be obtained from a single solution by means of transformations of the indicated group. In fact, such a situation obtains in the case of flow of fluid between two cylinders, in the problem of two-dimensional fluid surface waves, and also in the case of plane convection. An application of Theorem 1.1 to the problem of convection is given in Section 2.

1. A case of bifurcation in the presence of a multiple spectrum. We shall consider in a Banach space X, Equation of the form

$$x = K_\lambda x \tag{1.1}$$

Here, K_λ is a completely continuous operator in X, cancelling at zero. Let the Fréchet differential of operator K_λ at point $x = 0$ be $\lambda A x$, and λ_0 be the characteristic number of operator A. We assume that the following conditions are satisfied.

Condition 1.1. Operator K_λ is analytically determined in terms of x, λ in the region $(\|x\| < \rho;\ |\lambda - \lambda_0| < \gamma)$. Equation (1.1) may now be rewritten in the form

$$x = \lambda A x + \sum_{k=2}^{\infty} R_k x \tag{1.2}$$

where operator $R_k\, x = R_k\,(x, x, \ldots, x)$, k is linear and analytically dependent on λ

$$R_k x = \sum_{m=0}^{\infty} \mu^m R_{km} x, \qquad \mu = \lambda - \lambda_0 \tag{1.3}$$

Condition 1.2. Let L_g be the representation of a compact set G into the space of linear operators in X, i.e. L_g is a continuous operator-function on G, and let the following conditions be satisfied

$$L_{g_1 g_2} = L_{g_1} L_{g_2}, \qquad L_{g^{-1}} = L_g^{-1} \ (g, g_1, g_2 \in G) \tag{1.4}$$

We assume that Equation (1.1) is invariant with respect to the following transformations L_g:

$$L_g K_\lambda x = K_\lambda L_g x \qquad (x \in X;\ g \in G) \tag{1.5}$$

Expanding $K_\lambda(\rho x)$ into a power series of parameter ρ, and equating coefficients of like powers, we obtain from (1.5)

$$L_g A x = A L_g x; \qquad L_g R_k x = R_k L_g x, \qquad L_g R_{km} x = R_{km} L_g x \tag{1.6}$$

It follows from the first equality of (1.6) that the characteristic subspace X_0 of operator A, corresponding to the characteristic number λ_0, is invariant with respect to operators L_g.

Condition 1.3. We shall call the representation of L_g in X_0 complete, if for any pair of $\varphi', \varphi'' \in X_0$ we can indicate such $g \in G$ that

$$L_g \varphi' = \alpha \varphi'' \qquad (\alpha > 0) \tag{1.7}$$

We shall assume that the representation of L_g in X_0 is complete.

Condition 1.4. Let the rank of the characteristic number be unity, and the multiplicity coinciding in this case with the magnitude of X_0, be r. This signifies that basis $\varphi_0, \varphi_1, \ldots, \varphi_{r-1}$ can be indicated in X_0, and that there exists a system of eigenvectors of the conjugate operator A^*: $\psi_0, \psi_1, \ldots, \psi_{r-1}$ biorthogonal to $\{\varphi_k\}$.

It follows from (1.6) and (1.7) that such $g_k \in G$, for which $\varphi_k = L_{g_k}\varphi_0$ ($k = 1, 2, \ldots, r-1$) will be found.

Condition 1.5. We shall assume that when r is even, then a certain subspace E, consisting of vectors orthogonal to $\psi_1, \ldots, \psi_{r-1}$, and containing φ_0, is invariant with respect to operator K_λ.

Theorem 1.1. Let us assume that conditions 1.1 to 1.5 are fulfilled, and that the following inequality is true

$$\gamma = -(R_{20}{}^0(v, \varphi_0), \psi_0) - (R_{30}\varphi_0, \psi_0) > 0 \tag{1.8}$$

$$R_{20}{}^0(v, \varphi_0) = R_{20}(v, \varphi_0) + R_{20}(\varphi_0, v)$$

where vector v is defined as the solution of problem

$$v - \lambda_0 A v = R_{20}\varphi_0 - \sum_{k=0}^{r-1}(R_{20}\varphi_0, \psi_k)\varphi_k, \quad (v, \psi_0) = \cdots = (v, \psi_{r-1}) = 0 \tag{1.9}$$

Then:

a) new solutions of Equation (1.1) arise, when an increasing λ passes the value of λ_0: the spectrum lies to the right of point λ_0:

b) for every $\lambda > \lambda_0$, close to λ_0, there is one nonzero solution with an accuracy of the order of transformation L_g:

$$x = \sqrt{1/\lambda_0\gamma}\,\mu^{1/2}\varphi_0 + O(\mu)$$

Note. It will be seen from the subsequent analysis that, if conditions 1.2 and 1.3 are disregarded, and the assumption made that λ_0 is a prime eigen number, then, with condition (1.8) fulfilled, two nonzero solutions will occur

$$x_{1,2} = \mp\sqrt{1/\lambda_0\gamma}\,\mu^{1/2}\varphi_0 + O(\mu)$$

Proof. Any solution x' of Equation (1.1) may be presented in the form

$$x' = \sum_{k=0}^{r-1}\alpha_k'\varphi_k + y', \quad \alpha_k = (x, \psi_k), \quad (y, \psi_k) = 0 \quad (k = 0, 1, \ldots, r-1) \tag{1.10}$$

By virtue of 1.2, Equation (1.1) will have as a solution $x = L_g x'$ alongside with x' for any $g \in G$. In accordance with condition 1.3, element g can be so chosen that

$$L_g\left(\sum_{k=0}^{r-1}\alpha_k'\varphi_k\right) = \alpha\varphi_0 \qquad (\alpha > 0) \tag{1.11}$$

It follows from (1.10) and (1.11) that solution x is of the form

$$x = \alpha\varphi_0 + y, \quad (y, \psi_k) = 0 \quad (k = 0, 1, \ldots, r-1) \quad (\alpha > 0) \tag{1.12}$$

In fact, it follows from the first equality of (1.6) that $L_g{}^*$ commutates with A^*. Therefore, $L_g{}^*\psi_k$ is the eigenvector of operator A^*, consequently, by virtue of (1.10)

$$(y, \psi_k) = (L_g y', \psi_k) = (y', L_g{}^*\psi_k) = 0$$

Thus, any solution x' is obtained from a solution of form (1.12) by transformation L_g. The existence of a nonzero solution of Equation (1.1) follows immediately from the Krasnosel'skii's theorem [5] (in the case of an even r we change over to subspace

E, and utilize condition 1.5).

We shall now apply the Liapunov-Schmidt method to finding solutions of the form (1.12). Substituting (1.12) into (1.2), we obtain

$$y - \lambda_0 Ay = \mu\alpha / \lambda_0\varphi_0 + \mu Ay + \sum_{k=2}^{\infty} R_k(\alpha\varphi_0 + y) \equiv Ry, \quad \mu = \lambda - \lambda_0 \qquad (1.13)$$

Using the solvability conditions of Fredholm's equation, we rewrite (1.13) in the equivalent form as follows :

$$y - \lambda_0 Ay = Ry - \sum_{k=0}^{r-1} (Ry, \psi_k)\varphi_k, \qquad (Ry, \psi_k) = 0 \qquad (k = 0, 1, \ldots r, -1) \quad (1.14)$$

We shall look for small solutions of Equation (1.14) in the form of a power series

$$y = \sum_{p,\, q=0}^{\infty} \alpha^p \mu^q y_{pq}, \qquad y_{00} = 0, \qquad (y_{pq}, \psi_k) = 0 \qquad (k = 0, \ldots, r-1) \qquad (1.15)$$

Substituting (1.15) into (1.14), we deduce that $y_{10} = y_{01} = y_{11} = y_{02} = y_{12} = y_{03} = 0$, and for the determination of coefficients y_{20}, y_{30}, y_{21} we have the following equations

$$y_{20} - \lambda_0 Ay_{20} = P_0 R_{20}\varphi_0$$

$$y_{30} - \lambda_0 Ay_{30} = P_0 \{R_{20}{}^{\circ}(\varphi_0, y_{20}) + R_{30}\varphi_0\} \qquad (1.16)$$

$$y_{21} - \lambda_0 Ay_{21} = P_0 \{Ay_{20} + R_{21}\varphi_0\}$$

The projection operator P_0 is defined by Equation

$$P_0 x = x - \sum_{k=0}^{r-1} (x, \psi_k)\varphi_k, \qquad x \in X \qquad (1.17)$$

We thus have

$$y = y_{20}\alpha^2 + y_{30}\alpha^3 + y_{21}\alpha^2\mu + \ldots \qquad (1.18)$$

n which terms to powers higher than three have been omitted. Substituting (1.18) into he second equation of (1.14), we obtain for $k = 0$ the bifurcation equation in the form (1.19)

$$/ \lambda_0\alpha + \alpha^2 (R_{20}\varphi_0, \psi_0) + \alpha^3 [(R_{20}{}^{\circ}(\varphi_0, y_{20}), \psi_0) + (R_{30}\varphi_0, \psi_0)] + \mu\alpha^2 (R_{21}\varphi_0, \psi_0) + \ldots = 0$$

Here, terms to powers higher than three have again been omitted. It is clear that $(R_{20}\varphi_0, \psi_0) = 0$, as otherwise Equation (1.19) would have had only one nonzero solution α, whereas, from considerations at the beginning of this proof, it follows that there must be at least two roots (one positive, and one negative). Hence, Equation (1.19) can be written (see (1.18)) as

$$\frac{1}{\lambda_0}\mu\alpha - \gamma\alpha^3 + \mu\alpha^2 (R_{21}\varphi_0, \psi_0) + \ldots = 0 \qquad (1.20)$$

Using Newton's diagram [6], we deduce that Equation (1.20) has one (and only one) positive solution

$$\alpha = \sqrt{1 / \lambda_0\gamma}\,\mu^{1/2} + O(\mu) \qquad (1.21)$$

which exists for any small positive μ. The Theorem is proved.

We shall illustrate the application of this theorem by a simple example.

E x a m p l e . We shall consider the problem of finding a 2π-periodic solution of the ordinary differential equation

$$-u'' = \lambda u + uu' \qquad (1.22)$$

Converting operator $-d^2/dx^2$ by means of Green's operator A, we reduce Equation (1.22) to the form of (1.2), where

$$K_\lambda u = \lambda A u + R_{20} u, \quad R_{20} u = A\,(uu') \tag{1.23}$$

It is easy to show that operator K_λ is completely continuous in the Hilbert space H, in which the smooth 2π-periodic functions with a zero mean value in $(-\pi, \pi)$ are compact, while the scalar product is determined by Formula

$$(u_1, u_2)_H = \int_{-\pi}^{\pi} u_1' u_2' dx \tag{1.24}$$

Operator A is self-adjoint, rigorously positive, with eigen numbers and eigenfunctions which are

$$\lambda_{0k} = k^2, \quad \varphi_{0k} = \frac{1}{k\sqrt{\pi}} \sin kx, \quad \varphi_{1k} = \frac{1}{k\sqrt{\pi}} \cos kx \quad (k = 1, 2, \ldots) \tag{1.25}$$

Let G be a set of circle rotations. For $g \in G$ (g is the rotation by the angle g) we stipulate

$$L_g u = u\,(x + g) \tag{1.26}$$

Conditions 1.1 to 1.5 are readily verified, if we assume that E' in 1.5 is the subspace of odd functions from H. The value of γ is computed from (1.8). Equation (1.9) in this case is equivalent to the boundary value problem

$$-v'' = k^2 v + \frac{1}{\pi k^2} \sin kx \cos kx, \quad v\,(x + 2\pi) \equiv v\,(x), \quad v \perp \varphi_{0k}, \varphi_{1k} \tag{1.27}$$

Using Equations (1.27) and (1.8), we find

$$v = \frac{1}{6k^3\pi} \sin 2kx, \quad \gamma = -\int_{-\pi}^{\pi} (v\varphi_{0k}' + v'\varphi_{0k})\varphi_{0k}\, dx = \frac{1}{12\pi k^4} > 0 \tag{1.28}$$

In accordance with Theorem 1.1 a new solution occurs, when an increasing λ passes one of the values $\lambda_{0k} = k^2$ $(k = 1, 2, \ldots)$. It is

$$u_k = \frac{1}{k} \sqrt{12}\,(\lambda - \lambda_k)^{1/2} \sin kx + O\,(\lambda - \lambda_k) \tag{1.29}$$

All other solutions, bifurcating from the zero solution, are obtained from (1.29) by means of transformations (1.26).

2. Application to the problem of convection. Free convection in a fluid, filling a bounded space Ω, is described by the system

$$\nu \Delta \mathbf{v} - \nabla p = (\mathbf{v}, \nabla)\,\mathbf{v} + \beta\, T \mathbf{g}, \quad \chi \Delta T - \mathbf{v} \cdot \nabla T = c v_3, \quad \operatorname{div} \mathbf{v} = 0 \tag{2.1}$$

We shall assume that at the (sufficiently smooth) boundary S of the domain Ω the following boundary conditions are satisfied:

$$v = 0, \qquad T = 0 \tag{2.2}$$

Problem (2.1), (2.2) was reduced in [4 and 3] to the operator equation

$$\mathbf{v} = K\,(\mathbf{v}, c) = c\,A\mathbf{v} + R\mathbf{v} \tag{2.3}$$

in the Hilbert space H_1 of solenoidal vectors, vanishing at boundary S, and appertaining to $W_2^{(1)}$. This transformation is carried out in the following manner. Let $\mathbf{v} \in H_1$, $f\,(x) \in L_{6/5}\,(\Omega)$. We denote by $T = B_{\mathbf{v}} f$ the generalized solution of the boundary value problem

$$\chi \Delta T' - v \cdot \nabla T' = f, \qquad T'|_S = 0 \tag{2.4}$$

The second of Equations (2.1) yields now

$$\mathbf{v} = c B_{\mathbf{v}}\, v_3 = cM\,\mathbf{v} \tag{2.5}$$

The principle of compressed mapping makes it possible to obtain for small $\mathbf{v} \in H_1$, the following expansion

$$Mv = \sum_{k=1}^{\infty} M_k v, \qquad M_1 v = B_0 v_3, \qquad M_k v = B_0 (v \cdot \nabla M_{k-1} v) \qquad (2.6)$$

Operator M acts from the H_1 - into the H_2 -Hilbert space of functions vanishing at S, with scalar product

$$(T', T'')_{H_2} = \int_\Omega \nabla T' \cdot \nabla T'' \, dx$$

We next introduce operator L, which re-establishes the generalized solution of the linearized Navier-Stokes equations at their right-hand sides

$$\nu \triangle v - \nabla p = f, \qquad \operatorname{div} v = 0, \quad v|_S = 0, \qquad v = Lf \qquad (2.7)$$

It is now easy to proceed from (2. 1) to (2. 3), noting that

$$K(v, c) = L(v, \nabla) v + \beta c L(g M v)$$

$$Av = \beta L(g M_1 v), \qquad Rv = L(v, \nabla) v + \beta c \sum_{k=2}^{\infty} L(g M_k v) \qquad (2.8)$$

Operator A is completely continuous, self-adjoint, and rigorously positive [4] : its spectrum consists of positive characteristic numbers. We denote its smallest characteristic number by c_0, and shall consider it to be a prime number. The corresponding eigenvector will be denoted by

$$\varphi = c_0 A \varphi \qquad (2.9)$$

Assuming $\tau = c_0 B_0 \varphi_3$, we obtain the confirmation that the following equations are fulfilled

$$\nu \triangle \varphi - \nabla q = \beta \tau g, \qquad \operatorname{div} \varphi = 0$$
$$\chi \triangle \tau = c_0 \varphi_3, \qquad \tau|_S = 0, \quad \varphi|_S = 0 \qquad (2.10)$$

We introduce yet another vector $w \in H_1$, and function $\theta \in H_2$ as the solution of system

$$\nu \triangle w - \nabla p = (\varphi, \nabla)\varphi + \beta \theta g, \qquad \chi \triangle \theta = c_0 w_3 + \varphi \cdot \nabla \tau$$
$$\operatorname{div} w = 0, \qquad w|_S = 0, \qquad \theta|_S = 0, \qquad w \perp \varphi \qquad (2.11)$$

Lemma 2. 1. Problem (2. 1) is solvable and has a unique solution.

Proof. We proceed from (2. 11) to the operator equation in H_1. We have

$$0 = c_0 B_0 w_3 + B_0 (\varphi \cdot \nabla \tau) = c_0 (M_1 w + M_2 \varphi), \; w = c_0 A w + L(\varphi_1 \nabla) \varphi, \; w \perp \varphi \qquad (2.12)$$

If the second of Equations (2. 12) has a unique solution, then function θ is determined by the first equation. From the results of [7, 8 and 4] it follows that w, θ are as smooth as desired in Ω, provided that boundary S is sufficiently smooth.

There remains, thus, to verify the solvability condition of the equation defining w, i. e. the orthogonality of this equation free member to the eigenvector φ.

We have

$$(L(\varphi, \nabla)\varphi, \varphi)_{H_1} = -\int_\Omega \triangle L(\varphi, \nabla)\varphi \cdot \varphi \, dx = -\frac{1}{\nu} \int_\Omega [(\varphi, \nabla)\varphi + \nabla p] \cdot \varphi \, dx = 0 \quad (2.13)$$

The Lemma is proved.

We shall now calculate the value of γ, defined by (1. 8). In view of the self-adjointness of operator A, we have to assume that in (1. 8) $\psi_0 = \varphi_0 = \varphi$. We assume that

$$u = R_{20}(w, \varphi) + R_{30}\varphi \qquad (2.14)$$

With the aid of (2. 8) we obtain (2.15)

$$u = L[(w, \nabla)\varphi + (\varphi, \nabla) w] + \beta c_0 L[g B_0 (w \cdot \nabla B_0 \varphi_3 + \varphi \cdot \nabla B_0 w_3 + \varphi \cdot \nabla M_2 \varphi)]$$

In accordance with (1.8), (2.14) and (2.15) we have

$$\gamma = -(\mathbf{u}, \varphi)_{H_1} = \int_\Omega \Delta\mathbf{u}\cdot\varphi\,dx = \frac{1}{\nu}\int_\Omega (\varphi, \nabla)\mathbf{w}\cdot\varphi\,dx + $$
$$+\frac{\beta g c_0}{\nu}\int_\Omega \varphi_3 B_0(\mathbf{w}\cdot\nabla B_0\varphi_3 + \varphi\cdot\nabla B_0 w_3 + \varphi\cdot\nabla M_2\varphi)\,dx \qquad (2.16)$$

Here, and in the subsequent analysis, we shall use the following equalities readily deduced by integration by parts by taking advantage of the solenoidal properties of vectors $\mathbf{w}$, φ

$$\int_\Omega (\mathbf{w}, \nabla)\varphi\cdot\varphi\,dx = 0, \int_\Omega (\varphi, \nabla)\mathbf{w}\cdot\varphi\,dx = -\int_\Omega \mathbf{w}\cdot(\varphi, \nabla)\varphi\,dx. \quad (2.17)$$
$$\int_\Omega \tau\varphi\cdot\nabla\tau\,dx = 0$$

Making use of the self-adjointness of operator B_0, and of Equations (2.10), (2.12) and (2.17), we now reduce (2.16) to

$$\gamma = -\frac{1}{\nu}\int_\Omega \mathbf{w}\cdot(\varphi, \nabla)\varphi\,dx - \frac{\beta g}{\nu c_0}\int_\Omega \theta\varphi\cdot\nabla\tau\,dx \qquad (2.18)$$

Finally, the substitution in (2.18) for $(\varphi, \nabla)\varphi$, $\varphi\cdot\nabla\tau$ and χ of their expressions from (2.11), followed by integration by parts, yields

$$\gamma = \|\mathbf{w}\|^2_{H_1} + \frac{\beta g\chi}{\nu c_0}\|\theta\|^2_{H_2} + \frac{2\beta g}{\nu}\int_\Omega \theta w_3\,dx \qquad (2.19)$$

Lemma 2.2. Let $\mathbf{v}\in H_1$, $T\in H_2$. Then, inequality

$$J(\mathbf{v}, T) = \nu\|\mathbf{v}\|^2_{H_1} + \frac{\beta g\chi}{c_0}\|T\|^2_{H_2} + 2\beta g\int_\Omega T\mathbf{v}_3\,dx \geqslant 0 \qquad (2.20)$$

is valid. The equality is obtained only under condition

$$\mathbf{v} = \alpha\varphi, \qquad T = \alpha\tau, \qquad \alpha = \text{const} \qquad (2.21)$$

Here φ, τ is the eigensolution of problem (2.10).

Proof. According to the classical variational principle, equivalent to the first boundary value problem of Poisson's equation, functional $J(\mathbf{v}, T)$ with fixed $\mathbf{v}\in H_1$, reaches its minimum, when T is the solution of the following boundary value problem :

$$\chi\,\Delta T = c_0 v_3, \qquad T|_S = 0 \qquad (2.22)$$

or, in other words, at $T = c_0 B_0 v_3$. In this way, the inequality

$$J(\mathbf{v}, T) \geqslant J(\mathbf{v}, c_0 B_0 v_3) = \nu\|\mathbf{v}\|_{H_1}{}^2 - \beta g\chi c_0\|B_0 v_3\|_{H_2}{}^2 \qquad (2.23)$$

is satisfied.

However, for the smallest characteristic number c_0 of operator A in (2.9) the variational principle (see [4])

$$\frac{1}{c_0} = \max_{\mathbf{v}\in H_1}\frac{(A\mathbf{v}, \mathbf{v})_{H_1}}{\|\mathbf{v}\|_{H_1}{}^2} = \frac{\beta g\chi}{\nu}\max_{\mathbf{v}\in H_1}\frac{\|B_0 v_3\|_{H_2}{}^2}{\|\mathbf{v}\|_{H_1}{}^2} \qquad (2.24)$$

is valid. Its maximum is obtained only at $\mathbf{v} = \alpha\varphi$. It follows from (2.23) and (2.24) that the minimum value of functional $J(\mathbf{v}, T)$ is equal to zero, and is reached at $\mathbf{v} = \alpha\varphi$, $T = c_0 B_0 v_3 = \alpha\tau$. The Lemma is proved.

Since $\gamma = J(\mathbf{w}, \theta)/\nu$, it follows directly from Lemma 2.2 that $\gamma \geq 0$. We shall prove that γ is rigorously positive. As according to definition (2.11) $\mathbf{w}\perp\varphi$, it would follow from the equality $\mathbf{w} = \alpha\varphi$ that $\alpha = 0$, and consequently $\mathbf{w} = 0$, $\theta = 0$. Then, from (2.11) we have the following equalities :

$$(\varphi, \nabla)\varphi = -\nabla p, \qquad \varphi \cdot \nabla\tau = 0 \tag{2.25}$$

Lemma 2.3. Let φ, τ be a solution of system (2.10), and let the second of Equations (2.25) be fulfilled by it. Then, $\varphi = 0$, $\tau = 0$.

Proof. By virtue of (2.25) we have

$$0 = \int_\Omega x_3 \varphi \cdot \nabla\tau \, dx = -\int_\Omega \tau\varphi_3 \, dx \tag{2.26}$$

Multiplying now the first and the third equations of system (2,10), respectively, by φ and T, and integrating over the domain Ω using (2.26), we obtain

$$\nu \|\varphi\|_{H_1}^2 = -\beta g \int_\Omega \tau\varphi_3 dx = 0, \qquad \chi \|\tau\|_{H_2}^2 = -c_0 \int_\Omega \tau\varphi_3 \, dx = 0 \tag{2.27}$$

Thus, $\varphi = 0$, $\tau = 0$. The Lemma is proved.

It follows from Lemma 2.3 that, as shown by the preceding analysis, $\gamma > 0$. Using the Note to Theorem 1.1, we derive the following theorem.

Theorem 2.1. Let the smallest eigen number c_0 of the linearized problem (2.10) be a prime number. Then, for $c > c_0$ and sufficiently close to c_0, there exist two nonzero solutions of the operator equation (2.3), or (2.1)

$$\mathbf{v} = \mp \sqrt{(c - c_0)/c_0\gamma}\, \varphi + O(c - c_0), \qquad T = \mp \sqrt{(c - c_0)/c_0\gamma}\, \tau + O(c - c_0) \tag{2.28}$$

where the positive constant γ is defined by (2.18), or (2.19) (*).

Problem (2.1) has for any $c \leq c_0$ a unique solution $\mathbf{v} = 0$, $T = 0$, while for $c > c_0$ or close to c_0, it has exactly three: one zero solution and a pair of solutions (2.28). The whole of the interval (c_0, c_1), where c_1 is the second eigen number, appertains to the spectrum of Equation (2.3).

Proof. Only the last statement of this Theorem requires justification. We divide the proof into several Lemmas.

Lemma 2.4. All solutions of Equation (2.3) are contained within a sphere of space H_1 of radius m which depends only on domain Ω and parameters of (2.1).

Proof. For any $\mathbf{v} \in H_1$, $\varphi \in H_2$ the following inequalities of the kind of the Sobolev composition theorem are valid :

$$\|\mathbf{v}\|_{L_p} \leqslant m_p \|\mathbf{v}\|_{H_1}, \qquad \|\varphi\|_{L_p} \leqslant m_p \|\varphi\|_{H_2} \qquad (1 \leqslant p \leqslant 6) \tag{2.29}$$

We shall determine function $\psi(x)$, which is twice continuously differentiable in Ω and such that $\psi|_S = cx_3$. It can be further assumed that the following inequality is satisfied :

$$\|\psi\|_{L_4} \leqslant \varepsilon \tag{2.30}$$

where ε is arbitrarily small. It is easy to present function ψ in an explicit form, assuming that within the boundary strip it is a polynomial with respect to $\rho(x)$, that is, of the distance of point x from boundary S, and that outside of the boundary strip we have $\psi = 0$. We make the following substitution in Equations (2.1) :

$$T = T_0 + \psi - cx_3 \tag{2.31}$$

*) The possibility of the existence of a pair of secondary convective flows at supercritical values of temperature gradient was indicated in [9].

Function T_0 satisfies conditions

$$\chi\Delta T_0 - \mathbf{v}\cdot\nabla T_0 = -\chi\Delta\psi + \mathbf{v}\cdot\nabla\psi, \qquad T_0|_s = 0 \tag{2.32}$$

Multiplying (2.32) by T_0 and integrating over Ω, we obtain

$$\chi\|T_0\|_{H_2}^2 = -\chi\int_\Omega \nabla\psi\cdot\nabla T_0 dx - \int_\Omega \psi\mathbf{v}\cdot\nabla T_0 dx \tag{2.33}$$

Using Hölder's inequality and the composition Theorem (2.29), we derive from (2.33)

$$\chi\|T_0\|_{H_2} \leqslant \chi\|\nabla\psi\|_{L_2} + m_4\|\psi\|_{L_4}\cdot\|\mathbf{v}\|_{H_1} \tag{2.34}$$

We now multiply the first of Equations (2.1) by $\mathbf{v}$ and integrate over Ω. We obtain

$$\nu\|\mathbf{v}\|_{H_1}^2 = -\beta g\int_\Omega (T_0+\psi)\, v_3 dx \tag{2.35}$$

With the aid of (2.29) we derive from (2.35)

$$\nu\|\mathbf{v}\|_{H_1} \leqslant \beta g m_4^2\|T_0\|_{H_2} + \beta g m_2\|\psi\|_{L_2} \tag{2.36}$$

The required estimate now easily follows from (2.30), (2.35) and (2.36), if $\varepsilon = \chi\nu/2\beta g m_4^3$ is assumed. It is of the form

$$\|\mathbf{v}\|_{H_1} \leqslant \frac{2\beta g}{\nu}\left(m_4^2\|\nabla\psi\|_{L_2} + m_2\|\psi\|_{L_2}\right) = m \tag{2.37}$$

The Lemma is proved.

Lemma 2.5. Problem (2.1), (2.3) has a zero solution only when $c \leq c_0$.

Proof. Since the case of $c < c_0$ had been considered in [9 and 4], we shall assume that $c = c_0$. Multiplying the first of Equations (2.1) by $\mathbf{v}$ and the second by $\beta g T/c_0$, integrating over Ω, and adding, we obtain

$$J(\mathbf{v}, T) = 0 \tag{2.38}$$

In accordance with Lemma 2.2, it follows from (2.38) that $\mathbf{v} = \alpha\varphi$, $T = \alpha\tau$ (α = const). But then relationships (2.25) (in which a new function is substituted for p) must be fulfilled. And this, in accordance with Lemma 2.3, means that $\alpha = 0$. The Lemma is proved.

It follows from the general theory of bifurcation of operator equation solutions [6], that there exist such numbers μ_0, m_0 that for $|c-c_0| < \mu_0$ Equation (2.3) has no solutions in sphere $\|\mathbf{v}\|_{H_1} \leqslant m_0$ other than zero and the one given by (2.28). We introduce notations as follows:

$$\inf\|\mathbf{v} - K(\mathbf{v}, c_0)\|_{H_1} = \delta_1, \ \sup\|A\mathbf{v}\|_{H_1} = \delta_2 > 0 \tag{2.39}$$

$$(m_0 \leqslant \|\mathbf{v}\|_{H_1} < m_1,\ m_1 = \max m,\ c_0 \leqslant c \leqslant c_1)$$

Since $c = c_0$, Equation (2.3) has no nonzero solutions, and operator K is completely continuous, then $\delta_1 > 0$. Therefore, if

$$0 < c - c_0 < \delta_1/\delta_2 \tag{2.40}$$

Equation (2.3) cannot have any solutions outside sphere $\|\mathbf{v}\|_{H_1} \leqslant m_0$.

Indeed, in accordance with Lemma 2.4 there are no solutions outside the sphere $\|\mathbf{v}\|_{H_1} \leqslant m_1$, and there are none within layer $m_0 \leqslant \|\mathbf{v}\|_{H_1} \leqslant m_1$ by virtue of the simple estimate

$$\| \mathbf{v} - K(\mathbf{v}, c) \|_{H_1} \geqslant \| \mathbf{v} - K(\mathbf{v}, c_0) \|_{H_1} - (c - c_0) \| A\mathbf{v} \|_{H_1} \geqslant$$
$$\geqslant \delta_1 - (c - c_0)\delta_2 > 0 \quad (2.41)$$

Hence, with condition (2.40) fulfilled, all solutions of Equation (2.3) are contained within sphere $\| \mathbf{v} \|_{H_1} \leqslant m_0$, and there are exactly three of these in this sphere. Finally, it follows from Lemma 2.4 that the rotation of the vector field $\mathbf{v} - K(\mathbf{v}, c)$ on large spheres is $+1$. As the zero solution index for $c_0 < c < c_1$ is equal to -1, there must exist nonzero solutions (see [5 and 1]). Theorem 2.1 has been fully proved.

Example 1. Convection in a vertical cylinder. We shall indicate here, without detailed substantiation, a case in which the spectral problem (2.10) may be solved by the asymptotic method. Let space Ω be a cylinder with a vertical axis and normal cross section ω. We shall consider the case in which the cylinder height h is considerable.

We substitute in (2.10) variables $z = h\zeta/d$, where d is the diameter of section ω. We shall look for a solution of system (2.10), corresponding to the nth eigen number c_n in the form of a power series of the small parameter (*) $\epsilon = d/h$.

$$\varphi = \sum_{k=0}^{\infty} \varepsilon^k \varphi_k, \quad \tau = \sum_{k=0}^{\infty} \varepsilon^k \tau_k, \quad q = \sum_{k=-1}^{\infty} \varepsilon^k q_{k+1}, \quad c_n = \sum_{k=0}^{\infty} \varepsilon^k c_{nk} \quad (2.42)$$

Substituting (2.42) into (2.10), we readily deduce that

$$\varphi_{01} = \varphi_{02} = 0, \quad q_0 = q_0(\zeta) = a\zeta + \text{const}, \quad \varphi_{03} = w(x_1, x_2), \quad \tau_0 = \tau_0(x_1, x_2)$$

and obtain for the determination of c_{n0} the following spectral problem:

$$\nu \triangle w = a + \beta g \tau_0, \quad \chi \triangle \tau_0 = c_{n0} w, \quad w|_{S_0} = 0, \quad \tau_0|_{S_0} = 0, \quad \int_\omega w\, dx_1\, dx_2 = 0 \quad (2.43)$$

The last of conditions (2.43) implies that the flux velocity through the cross section ω is zero, which is the consequence of no-slip condition at the boundary S; S_0 is the boundary of section ω; a is an unknown constant.

We can obtain an explicit solution of problem (2.43) for a number of cases (for example when ω is a circle). We shall consider in greater detail the two-dimensional problem of convection in a rectangle. In this case $w = w(x)$, $\tau_0 = \tau_0(x)$, $x = x_1$, and problem (2.43) becomes ($d = 2$)

$$\nu w'' = a + \beta g \tau_0, \quad \chi \tau_0'' = c_{n0} w, \quad w = \tau = 0 \quad (x = \mp 1); \int_{-1}^{1} w(x)\, dx = 0 \quad (2.44)$$

Solutions of problem (2.44) are either even, or odd. Even solutions are of the form

$$w(x) = \cos\rho \cosh\rho x - \cosh\rho \cos\rho x \quad (2.45)$$

Function τ_0 is determined from the first of Equations (2.44), constant a by condition that $\tau_0(1) = 0$, and the corresponding eigenvalue is found from Equation

$$\tan\rho = \tanh\rho, \quad c = \varkappa\nu\rho^4 / \beta g, \quad \rho \neq 0 \quad (2.46)$$

For the odd solutions we have

$$a = 0, \quad w = \beta g \sin\rho x, \quad \tau = -\nu\rho^2 \sin\rho x, \quad \rho = k\pi \quad (k = 1, 2, \ldots) \quad (2.47)$$

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*) Expansion (2.42) is valid at some distance from the cylinder bottom $\zeta = 0.1$, where boundary layer phenomena occur. It is important to note, that c_{n0} is determined independently of the construction of boundary value solutions.

It follows from (2.45) to (2.47) that all eigen numbers c_{0n} are prime numbers. Indeed, Equation (2.46) has a single root within each segment $(k\pi, (2k+1)\pi/2)$ $(k = 1, 2, \ldots)$ and no other roots whatsoever. In particular, its smallest root is $\rho_1 = 3.9264$. But, $c_n \to c_{0n}$ when $\epsilon \to 0$ (we stress the uneven convergence with respect to n). Therefore, all of the first eigen numbers c_0, c_1, ..., c_k (k is any arbitrary given number) are prime numbers, provided that ϵ is sufficiently small.

Hence, Theorem 2.1 is applicable to problems of convection in a vertical rectangular vessel of a height considerably greater than its width.

Note. A similar result is also easily obtained in the case of a cylinder of circular cross section (relevant computations were carried out in [10], p. 50). The first eigen number of problem (2.44) is probably always a prime number. Subsequent eigen numbers may, however, be multiple numbers, as was shown on the example of a rectangular ω. Multiple eigen numbers occur, however, only rarely, as in the periodic problem considered in [2 and 3], and only with special dimensional relationships.

Example 2. Two-dimensional convection in a horizontal channel. We shall consider the two-dimensional problem (2.1) in a strip defined by $0 \leq z \leq h$. We assume that velocity $\mathbf{v}$ is periodic with respect to $x = x_1$, with period $2\pi/\alpha_0$, and that the flux velocity through the cross section is zero. As was shown in [3], there exist for all values of α_0 double eigenvalues of the relevant linearized problem (2.10), with the exception of a certain denumerable set, to which correspond the following eigenvalues

$$\tau_1(x, z) = \pi(z)\cos\alpha x, \quad \tau_2(x, z) = \pi(z)\sin\alpha x, \quad \varphi_i = L(\beta\tau_i \mathbf{g}) \ (i = 1, 2) \qquad (2.48)$$

The problem is invariant with respect to shear along the x-axis. We introduce operators L_g and $\mathbf{L}_g$

$$\tau_g = L_g\tau(x, z) = \tau(x + g, z), \quad \varphi_g = \varphi(x + g, z) = \mathbf{L}_g\varphi \qquad (2.49)$$

For functions τ and vectors φ periodic with respect to x, parameter g may be considered as an element of set G of circle rotations. Conditions 1.1 to 1.5 of Theorem 1.1 are evidently satisfied. The validity of condition (1.8) follows from Theorem 2.1.

Hence, a two-dimensional convection in a channel is uniquely defined by a period $2\pi/\alpha_0$ (with an accuracy of the order of shear along the x-axis) for all α_0, except of the case of a denumerable set.

Example 3. Cellular convection in a layer. We shall now consider the problem of a twofold periodic, or hexagonal convection in a horizontal layer of fluid heated from below [3]. Using Theorem 2.1 in conditions similar to those considered in [3], we find that when the temperature gradient passes through the first critical value, a pair of solutions occurs. We would remind that we are considering here flows which satisfy conditions of periodicity (or hexagonality), as well as certain supplementary conditions as regards evenness. We note that it is easy to show on this example that the bifurcation proceeds in an analogous manner, not only for the first eigenvalues, but also for all subsequent eigenvalues. However, in such cases unstable solutions occur.

It may be further pointed out that here secondary flows differ insignificantly: they are obtained one from another by shifting in plane x_1x_2.

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